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#### Recommended Citation

McLaughlin, J., & Zimmer, P. (2009). Some Implications of the WP-Bailey Tree. *Advances in Applied Mathematics*, 43(2), 162-175. Retrieved from [https://digitalcommons.wcupa.edu/math\\_facpub/51](https://digitalcommons.wcupa.edu/math_facpub/51)

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# SOME IMPLICATIONS OF THE WP-BAILEY TREE

JAMES MC LAUGHLIN AND PETER ZIMMER

ABSTRACT. We consider a special case of a WP-Bailey chain of George Andrews, and use it to derive a number of curious transformations of basic hypergeometric series.

We also derive two new WP-Bailey pairs, and use them to derive some new transformations for basic hypergeometric series.

Finally, we briefly consider the implications of WP-Bailey pairs  $(\alpha_n(a, k), \beta_n(a, k))$ , in which  $\alpha_n(a, k)$  is independent of  $k$ , for generalizations of identities of the Rogers-Ramanujan type.

## 1. INTRODUCTION

Andrews, building on prior work of Bressoud [5] and Singh [9], in [1] defined a *WP-Bailey pair* to be a pair of sequences  $(\alpha_n(a, k), \beta_n(a, k))$  satisfying

$$(1.1) \quad \begin{aligned} \beta_n(a, k) &= \sum_{j=0}^n \frac{(k/a)_{n-j} (k)_{n+j}}{(q)_{n-j} (aq)_{n+j}} \alpha_j(a, k) \\ &= \frac{(k/a, k; q)_n}{(aq, q; q)_n} \sum_{j=0}^n \frac{(q^{-n})_j (kq^n)_j}{(aq^{1-n}/k)_j (aq^{n+1})_j} \left(\frac{qa}{k}\right)^j \alpha_j(a, k). \end{aligned}$$

Andrews also showed in [1] that there were two distinct ways to construct new WP-Bailey pairs from a given pair. If  $(\alpha_n(a, k), \beta_n(a, k))$  satisfy (1.1), then so do  $(\alpha'_n(a, k), \beta'_n(a, k))$  and  $(\tilde{\alpha}_n(a, k), \tilde{\beta}_n(a, k))$ , where

$$(1.2) \quad \begin{aligned} \alpha'_n(a, k) &= \frac{(\rho_1, \rho_2)_n}{(aq/\rho_1, aq/\rho_2)_n} \left(\frac{k}{c}\right)^n \alpha_n(a, c), \\ \beta'_n(a, k) &= \frac{(k\rho_1/a, k\rho_2/a)_n}{(aq/\rho_1, aq/\rho_2)_n} \\ &\quad \times \sum_{j=0}^n \frac{(1 - cq^{2j})(\rho_1, \rho_2)_j (k/c)_{n-j} (k)_{n+j}}{(1 - c)(k\rho_1/a, k\rho_2/a)_n (q)_{n-j} (qc)_{n+j}} \left(\frac{k}{c}\right)^j \beta_j(a, c), \end{aligned}$$

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*Date:* November 14, 2008.

*2000 Mathematics Subject Classification.* Primary: 33D15. Secondary: 11B65, 05A19.

*Key words and phrases.* Q-Series, Rogers-Ramanujan Type Identities, Bailey chains, finite Rogers-Ramanujan identities, WP-Bailey pairs.

with  $c = k\rho_1\rho_2/aq$  for the pair above, and

$$(1.3) \quad \begin{aligned} \tilde{\alpha}_n(a, k) &= \frac{(qa^2/k)_{2n}}{(k)_{2n}} \left( \frac{k^2}{qa^2} \right)^n \alpha_n \left( a, \frac{qa^2}{k} \right), \\ \tilde{\beta}_n(a, k) &= \sum_{j=0}^n \frac{(k^2/qa^2)_{n-j}}{(q)_{n-j}} \left( \frac{k^2}{qa^2} \right)^j \beta_j \left( a, \frac{qa^2}{k} \right). \end{aligned}$$

These two constructions allow a “tree” of WP-Bailey pairs to be generated from a single WP-Bailey pair. Andrews and Berkovich [2] further investigated these two branches of the WP-Bailey tree, in the process deriving many new transformations for basic hypergeometric series. Spiridonov [11] derived an elliptic generalization of Andrews first WP-Bailey chain, and Warnaar [14]<sup>1</sup> added four new branches to the WP-Bailey tree, two of which had generalizations to the elliptic level. More recently, Liu and Ma [7] introduced the idea of a general WP-Bailey chain (as a solution to a system of linear equations), and added one new branch to the WP-Bailey tree.

In the present paper, we derive two new WP-Bailey pairs, one of them restricted in the sense that it is necessary to set  $k = q$ . We then insert these in some of the WP-Bailey chains listed above to derive new transformations of basic hypergeometric series.

We also consider a special case ( $k = aq$ ) of the first WP-Bailey chain of Andrews, and show how it leads some unusual transformations of series.

We also briefly consider the special case of a WP-Bailey pair  $(\alpha_n(a), \beta_n(a, k))$ , where the  $\alpha_n$  are independent of  $k$ . We show how a such pair may give rise to a generalization of a Slater-type identity deriving from the standard Bailey pair  $(\alpha_n(a), \beta_n(a, 0))$ .

## 2. FINITE BASIC HYPERGEOMETRIC IDENTITIES DERIVING FROM “TRIVIAL” WP-BAILEY PAIRS

In this section we derive some unusual transformations from (1.2) by inserting “trivial” WP-Bailey pairs (see below for the definition).

We reformulate the constructions at (1.2) and (for later use) (1.3) as transformations relating the original WP-Bailey pair  $(\alpha_n(a, k), \beta_n(a, k))$ . We first recall the following elementary transformation:

$$(2.1) \quad (a; q)_{n-r} = \frac{q^{r(r+1)/2}(a; q)_n}{(q^{1-n}/a; q)_r(-aq^n)^r}.$$

**Theorem 1.** *If  $(\alpha_n(a, k), \beta_n(a, k))$  satisfy*

$$(2.2) \quad \beta_n(a, k) = \sum_{j=0}^n \frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}} \alpha_j(a, k),$$

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<sup>1</sup>In a note added after submitting the paper [14], Warnaar remarks that he had discovered many more transformations for basic and elliptic WP-Bailey pairs, and gives two further examples of chains that hold at the elliptic level.

then

$$\begin{aligned}
 (2.3) \quad & \sum_{n=0}^N \frac{(1 - kq^{2n})(\rho_1, \rho_2, kaq^{N+1}/\rho_1\rho_2, q^{-N}; q)_n}{(1 - k)(kq/\rho_1, kq/\rho_2, \rho_1\rho_2q^{-N}/a, kq^{1+N}; q)_n} q^n \beta_n(a, k) \\
 &= \frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2; q)_N}{(kq/\rho_1, kq/\rho_2, aq/\rho_1\rho_2, aq; q)_N} \\
 &\quad \times \sum_{n=0}^N \frac{(\rho_1, \rho_2, kaq^{N+1}/\rho_1\rho_2, q^{-N}; q)_n}{(aq/\rho_1, aq/\rho_2, aq^{1+N}, \rho_1\rho_2q^{-N}/k; q)_n} \left(\frac{aq}{k}\right)^n \alpha_n(a, k),
 \end{aligned}$$

and

$$\begin{aligned}
 (2.4) \quad & \sum_{n=0}^N \frac{(q^{-N}; q)_n}{(q^{-N}k^2/a^2; q)_n} q^n \beta_n(a, k) = \frac{(qa/k, qa^2/k; q)_N}{(qa, qa^2/k^2; q)_N} \\
 &\quad \times \sum_{n=0}^N \frac{(q^{N+1}a^2/k, q^{-N}; q)_n (k; q)_{2n}}{(aq^{1+N}, q^{-N}k/a; q)_n (qa^2/k; q)_{2n}} \left(\frac{aq}{k}\right)^n \alpha_n(a, k).
 \end{aligned}$$

*Proof.* The identity at (2.3) follows from (1.2), after substituting for  $\alpha'_n(a, k)$  in (1.1), employing (2.1), then setting the two expressions for  $\beta'_n(a, k)$  equal, and finally replacing  $c$  with  $k$ . The identity at (2.4) follows from (1.3), after setting  $k = qa^2/c$ , using (2.1), and finally replacing  $c$  with  $k$ .  $\square$

For later use we also note the following corollary, which is immediate upon letting  $N \rightarrow \infty$ .

**Corollary 1.** *If  $(\alpha_n(a, k), \beta_n(a, k))$  satisfy*

$$\beta_n(a, k) = \sum_{j=0}^n \frac{(k/a)_{n-j} (k)_{n+j}}{(q)_{n-j} (aq)_{n+j}} \alpha_j(a, k),$$

then

$$\begin{aligned}
 (2.5) \quad & \sum_{n=0}^{\infty} \frac{(1 - kq^{2n})(\rho_1, \rho_2; q)_n}{(1 - k)(kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n(a, k) = \\
 & \frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2; q)_{\infty}}{(kq/\rho_1, kq/\rho_2, aq/\rho_1\rho_2, aq; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a, k),
 \end{aligned}$$

and

$$\begin{aligned}
 (2.6) \quad & \sum_{n=0}^{\infty} \left(\frac{qa^2}{k^2}\right)^n \beta_n(a, k) \\
 &= \frac{(qa/k, qa^2/k; q)_{\infty}}{(qa, qa^2/k^2; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(k; q)_{2n}}{(qa^2/k; q)_{2n}} \left(\frac{qa^2}{k^2}\right)^n \alpha_n(a, k).
 \end{aligned}$$

Remark: At several places throughout the paper we replace  $a$  with  $x/q$ ,  $\rho_1$  with  $y$  and  $\rho_2$  with  $z$ , in order to more easily make comparisons with a transformation due to Bailey (4.4) later.

We now consider some applications of the following corollary.

**Corollary 2.** *Let  $N$  be a positive integer. Suppose the sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  are related by*

$$(2.7) \quad \beta_n = \sum_{r=0}^n \alpha_r.$$

Then

$$(2.8) \quad \sum_{n=0}^N \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2 q^N / yz, q^{-N}; q)_n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_n} q^n \beta_n \\ = \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ \times \sum_{n=0}^N \frac{(y, z, k^2 q^N / yz, q^{-N}; q)_n}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n} \alpha_n.$$

*Proof.* Let  $\rho_1 = y$ ,  $\rho_2 = z$  and  $a = k/q$  in (2.2) and (2.3) in Theorem 1.  $\square$

We could reasonably call pairs  $(\alpha_n, \beta_n)$  satisfying (2.7) *trivial WP-Bailey pairs*. It is interesting that the simple condition relating  $\alpha_n$  and  $\beta_n$  at (2.7) should have several non-trivial consequences. Before noticing the result in Corollary 2, we had first remarked in [8] the result that derives from it by letting  $N \rightarrow \infty$  (found by another method), so in fact many of the results in [8] has a finite counterpart. We give some examples.

**Corollary 3.** *Let  $N$  be a positive integer. Then*

$$(2.9) \quad \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2 q^N / yz, q^{-N}; q)_{2n} q^{2n}}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_{2n}} \\ = \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ \times \sum_{n=0}^N \frac{(y, z, k^2 q^N / yz, q^{-N}; q)_n}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n} (-1)^n.$$

*Proof.* Let  $\alpha_n = (-1)^n$  in Corollary 2.  $\square$

Remark: The left side in (2.9) above could be expressed (see [6, page 39] for an explanation of the notation) as

$${}_{12}W_{11} \left( k; y, yq, z, zq, \frac{k^2 q^N}{yz}, \frac{k^2 q^{1+N}}{yz}, q^{-N}, q^{1-N}, q^2; q^2; q^2 \right),$$

where the additional  $q$ -products are inserted here to give the series the form of a  ${}_{r+1}W_r$  series.

**Corollary 4.** *Let  $N$  be a positive integer. Then*

$$\begin{aligned} \sum_{n=0}^N \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2q^N/yz, q^{-N}; q)_n (n+1) q^n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_n} \\ = \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ \times \sum_{n=0}^N \frac{(y, z, k^2q^N/yz, q^{-N}; q)_n}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n}. \end{aligned}$$

*Proof.* Let  $\alpha_n = 1$  in Corollary 2. □

**Corollary 5.**

$$\begin{aligned} (2.10) \quad {}_8\phi_7 \left[ \begin{matrix} q\sqrt{k}, -q\sqrt{k}, y, z, aq, bq, k^2q^N/yz, q^{-N} \\ \sqrt{k}, -\sqrt{k}, qk/y, qk/z, (a+b-1)q, yzq^{1-N}/k, kq^{1+N}; q, q \end{matrix} \right] \\ = \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ \times {}_6\phi_5 \left[ \begin{matrix} y, z, a, b, k^2q^N/yz, q^{-N} \\ k/y, k/z, (a+b-1)q, kq^N, yzq^{-N}/k; q, q^2 \end{matrix} \right]. \end{aligned}$$

*Proof.* In Corollary 2, define  $\alpha_0 = 1$ , and for  $n > 0$ ,

$$\alpha_n = \frac{(aq, bq; q)_n}{((a+b-1)q, q; q)_n} - \frac{(aq, bq; q)_{n-1}}{((a+b-1)q, q; q)_{n-1}} = \frac{(a, b; q)_n q^{2n}}{((a+b-1)q, q; q)_n}.$$

The sum for  $\beta_n$  is easily seen to telescope to give

$$\beta_n = \frac{(aq, bq; q)_n}{((a+b-1)q, q; q)_n}.$$

and the result follows. □

**Corollary 6.**

$$\begin{aligned} (2.11) \quad {}_8\phi_7 \left[ \begin{matrix} q\sqrt{k}, -q\sqrt{k}, y, z, aq, bq, k^2q^N/yz, q^{-N} \\ \sqrt{k}, -\sqrt{k}, qk/y, qk/z, abq, yzq^{1-N}/k, kq^{1+N}; q, q \end{matrix} \right] \\ = \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ \times {}_6\phi_5 \left[ \begin{matrix} y, z, a, b, k^2q^N/yz, q^{-N} \\ k/y, k/z, abq, kq^N, yzq^{-N}/k; q, q \end{matrix} \right]. \end{aligned}$$

*Proof.* This time, in Corollary 2, define  $\alpha_0 = 1$ , and for  $n > 0$ ,

$$\alpha_n = \frac{(aq, bq; q)_n}{(abq, q; q)_n} - \frac{(aq, bq; q)_{n-1}}{(abq, q; q)_{n-1}} = \frac{(a, b; q)_n q^n}{(abq, q; q)_n}.$$

The result follows as above. □

**Corollary 7.** *Let  $N$  and  $m$  be positive integers and let  $p$  be an integer. Then*

$$\begin{aligned} & \sum_{n=0}^N \frac{(q\sqrt{k}, -q\sqrt{k}, y, z, k^2q^N/yz, q^{-N}; q)_n q^{(mn^2+(p+2)n)/2}}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z, yzq^{1-N}/k, kq^{1+N}; q)_n (-q^{(m+p)/2}; q^m)_n} \\ &= \frac{(1-k/y)(1-k/z)(1-kq^N)(1-kq^N/yz)}{(1-k)(1-k/yz)(1-kq^N/y)(1-kq^N/z)} \\ & \times \left( 1 - q^{(m-p)/2} \sum_{n=1}^N \frac{(y, z, k^2q^N/yz, q^{-N}; q)_n q^{(mn^2+(p-2m)n)/2}}{(k/y, k/z, kq^N, yzq^{-N}/k; q)_n (-q^{(m+p)/2}; q^m)_n} \right). \end{aligned}$$

*Proof.* In Corollary 2 set  $\alpha_0 = 1$  and, for  $n > 0$ ,

$$\alpha_n = \frac{q^{(mn^2+pn)/2}}{(-q^{(m+p)/2}; q^m)_n} - \frac{q^{(m(n-1)^2+p(n-1))/2}}{(-q^{(m+p)/2}; q^m)_{n-1}} = -q^{(m-p)/2} \frac{q^{(mn^2+(p-2m)n)/2}}{(-q^{(m+p)/2}; q^m)_n}.$$

□

**Corollary 8.** *Let  $P, p, Q, q, R, a, b, c, k, y$  and  $z$  be complex numbers such that none of the denominators below vanish. Then*

$$\begin{aligned} (2.12) \quad & \sum_{n=0}^N \frac{\left(q\sqrt{k}, -q\sqrt{k}, y, z, \frac{k^2q^N}{yz}, q^{-N}; q\right)_n}{\left(\sqrt{k}, -\sqrt{k}, \frac{qk}{y}, \frac{qk}{z}, \frac{yzq^{1-N}}{k}, kq^{1+N}; q\right)_n} \\ & \times \frac{(ap^2; p^2)_n (bP^2; P^2)_n (cR^2; R^2)_n \left(\frac{aQ^2}{bc}; Q^2\right)_n q^n}{\left(\frac{PQR}{p}, \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}, \frac{pPQ}{R}\right)_n \left(\frac{apQR}{bP}, \frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q}, \frac{pPR}{Q}\right)_n} \\ &= \frac{\left(1 - \frac{k}{y}\right) \left(1 - \frac{k}{z}\right) (1 - kq^N) \left(1 - \frac{kq^N}{yz}\right)}{(1-k) \left(1 - \frac{k}{yz}\right) \left(1 - \frac{kq^N}{y}\right) \left(1 - \frac{kq^N}{z}\right)} \sum_{n=0}^N \frac{\left(y, z, \frac{k^2q^N}{yz}, q^{-N}; q\right)_n}{\left(\frac{k}{y}, \frac{k}{z}, kq^N, \frac{yzq^{-N}}{k}; q\right)_n} \\ & \times \frac{(1 - ap^n P^n Q^n R^n) \left(1 - b \frac{p^n P^n}{Q^n R^n}\right) \left(1 - \frac{P^n Q^n}{cp^n R^n}\right) \left(1 - \frac{ap^n Q^n}{bc P^n R^n}\right)}{(1-a)(1-b) \left(1 - \frac{1}{c}\right) \left(1 - \frac{a}{bc}\right)} \\ & \times \frac{(a; p^2)_n (b; P^2)_n (c; R^2)_n \left(\frac{a}{bc}; Q^2\right)_n R^{2n}}{\left(\frac{PQR}{p}, \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}, \frac{pPQ}{R}\right)_n \left(\frac{apQR}{bP}, \frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q}, \frac{pPR}{Q}\right)_n}; \\ (2.13) \quad & \sum_{n=0}^N \frac{\left(q\sqrt{k}, -q\sqrt{k}, y, z, \frac{k^2q^N}{yz}, q^{-N}; q\right)_n}{\left(\sqrt{k}, -\sqrt{k}, \frac{qk}{y}, \frac{qk}{z}, \frac{yzq^{1-N}}{k}, kq^{1+N}; q\right)_n} \frac{(aq^m, bq^m, cq^m, \frac{aq^m}{bc}; q^m)_n}{\left(\frac{a}{c}q^m, \frac{a}{b}q^m, bcq^m, q^m; q^m\right)_n} q^n \\ &= \frac{\left(1 - \frac{k}{y}\right) \left(1 - \frac{k}{z}\right) (1 - kq^N) \left(1 - \frac{kq^N}{yz}\right)}{(1-k) \left(1 - \frac{k}{yz}\right) \left(1 - \frac{kq^N}{y}\right) \left(1 - \frac{kq^N}{z}\right)} \times \end{aligned}$$

$$\sum_{n=0}^N \frac{\left(y, z, \frac{k^2 q^N}{yz}, q^{-N}; q\right)_n}{\left(\frac{k}{y}, \frac{k}{z}, kq^N, \frac{yzq^{-N}}{k}; q\right)_n} \frac{(q^m \sqrt{a}, -q^m \sqrt{a}, a, b, c, \frac{a}{bc}; q^m)_n q^{mn}}{(\sqrt{a}, -\sqrt{a}, \frac{a}{c} q^m, \frac{a}{b} q^m, bcq^m, q^m; q^m)_n}.$$

*Proof.* We use the special case  $m = 0$ ,  $d = 1$  of the identity of Subbarao and Verma labeled (2.2) in [12], namely,

$$(2.14) \quad \sum_{k=0}^n \frac{(1 - ap^k P^k Q^k R^k) \left(1 - b \frac{p^k P^k}{Q^k R^k}\right) \left(1 - \frac{P^k Q^k}{cp^k R^k}\right) \left(1 - \frac{ap^k Q^k}{bcP^k R^k}\right)}{(1-a)(1-b) \left(1 - \frac{1}{c}\right) \left(1 - \frac{a}{bc}\right)} \\ \times \frac{(a; p^2)_k (b; P^2)_k (c; R^2)_k \left(\frac{a}{bc}; Q^2\right)_k}{\left(\frac{PQR}{p}; \frac{PQR}{p}\right)_k \left(\frac{apPQ}{cR}; \frac{pPQ}{R}\right)_k \left(\frac{apQR}{bP}; \frac{pQR}{P}\right)_k \left(\frac{bcpPR}{Q}; \frac{pPR}{Q}\right)_k} R^{2k} \\ = \frac{(ap^2; p^2)_n (bP^2; P^2)_n (cR^2; R^2)_n \left(\frac{aQ^2}{bc}; Q^2\right)_n}{\left(\frac{PQR}{p}; \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}; \frac{pPQ}{R}\right)_n \left(\frac{apQR}{bP}; \frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q}; \frac{pPR}{Q}\right)_n},$$

and then, in (2.8) above, let  $\alpha_i$  be the  $i$ -th term in the sum above, and let  $\beta_n$  be the quantity on the right side above.

The identity at (2.13) follows upon setting  $P = Q = p = R = q^{m/2}$  and simplifying.  $\square$

Apart from the first chain of Andrews, there is only one other WP-Bailey chain, of those seven alluded to in the introduction, that leads to a non-trivial transformation similar to that in Corollary 2 (the consequences of letting  $k = aq$  in the other chains being entirely trivial). This is the third chain of Warnaar (Theorem 2.5 in [14]). This chain implies that if  $(\alpha_n(a, k), \beta_n(a, k))$  satisfy (1.1), then

$$(2.15) \quad \sum_{j=0}^n \frac{1 + aq^{2j}}{1 + a} \frac{(m/a; q^2)_{n-j} (am; q^2)_{n+j}}{(q^2; q^2)_{n-j} (a^2 q^2; q^2)_{n+j}} q^{-j} \alpha_j(a, m) \\ = q^{-n} \frac{(-mq; q)_{2n}}{(-a; q)_{2n}} \\ \times \sum_{j=0}^n \frac{1 - mq^{2j}}{1 - m} \frac{(a/m; q^2)_{n-j} (am; q^2)_{n+j}}{(q^2; q^2)_{n-j} (m^2 q^2; q^2)_{n+j}} \left(\frac{m}{a}\right)^{n-j} \beta_j(a, m).$$

Upon setting  $m = aq$ , we get that if  $\beta_n = \sum_{j=0}^n \alpha_j$ , then

$$(2.16) \quad \sum_{j=0}^n \frac{1 + aq^{2j}}{1 + a} \frac{(q; q^2)_{n-j} (a^2 q; q^2)_{n+j}}{(q^2; q^2)_{n-j} (a^2 q^2; q^2)_{n+j}} q^{-j} \alpha_j \\ = \frac{(1 + aq^{2n})(1 + aq^{2n+1})}{(1 + a)(1 + aq)} \sum_{j=0}^n \frac{1 - aq^{2j+1}}{1 - aq} \frac{(1/q; q^2)_{n-j} (a^2 q; q^2)_{n+j}}{(q^2; q^2)_{n-j} (a^2 q^4; q^2)_{n+j}} q^{-j} \beta_j.$$



However, we do not pursue the consequences of this transformation further here.

### 3. NEW WP-BAILEY PAIRS

We next exhibit two new WP-Bailey pairs.

**Lemma 1.** *The pair  $(\alpha_n^{(1)}(a, k), \beta_n^{(1)}(a, k))$  is a WP-Bailey pair, where*

$$(3.1) \quad \begin{aligned} \alpha_n^{(1)}(a, k) &= \frac{(qa^2/k^2; q)_n}{(q; q)_n} \left(\frac{k}{a}\right)^n, \\ \beta_n^{(1)}(a, k) &= \frac{(qa/k, k; q)_n (k^2/a; q)_{2n}}{(k^2/a, q; q)_n (aq; q)_{2n}}. \end{aligned}$$

*Proof.*

$$\begin{aligned} \sum_{j=0}^n \frac{(k/a)_{n-j} (k)_{n+j}}{(q)_{n-j} (aq)_{n+j}} \alpha_j^{(1)}(a, k) &= \frac{(k/a, k; q)_n}{(aq, q; q)_n} \sum_{j=0}^n \frac{(q^{-n}, kq^n; q)_j}{(aq^{1-n}/k, aq^{n+1}; q)_j} \left(\frac{qa}{k}\right)^j \alpha_j^{(1)}(a, k) \\ &= \frac{(k/a, k; q)_n}{(aq, q; q)_n} \sum_{j=0}^n \frac{(q^{-n}, kq^n, qa^2/k^2; q)_j}{(aq^{1-n}/k, aq^{n+1}, q; q)_j} q^j \\ &= \frac{(k/a, k; q)_n}{(aq, q; q)_n} \frac{(aq/k, k^2q^n/a; q)_n}{(aq^{n+1}, k/a; q)_n} \\ &= \frac{(qa/k, k; q)_n (k^2/a; q)_{2n}}{(k^2/a, q, q)_n (aq, q)_{2n}} = \beta_n^{(1)}(a, k). \end{aligned}$$

The third equality follows from the  $q$ -Pfaff-Saalschütz sum,

$$(3.2) \quad {}_3\phi_2 \left[ \begin{matrix} a, b, q^{-n} \\ c, abq^{1-n}/c \end{matrix}; q, q \right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}.$$

□

**Lemma 2.** *The pair  $(\alpha_n^{(2)}(a, q), \beta_n^{(2)}(a, q))$  is a WP-Bailey pair, where*

$$(3.3) \quad \begin{aligned} \alpha_n^{(2)}(a, q) &= \frac{(a, q\sqrt{a}, -q\sqrt{a}, d, q/d, -a; q)_n}{(\sqrt{a}, -\sqrt{a}, aq/d, ad, -q, q; q)_n} (-1)^n, \\ \beta_n^{(2)}(a, q) &= \begin{cases} \frac{(q^2/ad, dq/a; q^2)_{n/2}}{(adq, aq^2/d; q^2)_{n/2}}, & n \text{ even}, \\ -a \frac{(q/ad, d/a; q^2)_{(n+1)/2}}{(ad, aq/d; q^2)_{(n+1)/2}}, & n \text{ odd}. \end{cases} \end{aligned}$$

Remark: Note that this pair is restricted in that it is necessary to set  $k = q$  for (1.1) to hold.

*Proof.*

$$\begin{aligned}
& \sum_{j=0}^n \frac{(q/a)_{n-j}(q)_{n+j}}{(q)_{n-j}(aq)_{n+j}} \alpha_j^{(2)}(a, q) \\
&= \frac{(q/a; q)_n}{(aq; q)_n} \sum_{j=0}^n \frac{(q^{-n}, q^{n+1}; q)_j}{(aq^{-n}, aq^{n+1}; q)_j} a^j \alpha_j^{(2)}(a, q) \\
&= \frac{(q/a; q)_n}{(aq; q)_n} \sum_{j=0}^n \frac{(q^{-n}, q^{n+1}, a, q\sqrt{a}, -q\sqrt{a}, d, \frac{q}{d}, -a; q)_j}{(aq^{-n}, aq^{n+1}, \sqrt{a}, -\sqrt{a}, \frac{qa}{d}, ad, -q, q; q)_j} (-a)^j \\
&= \frac{(q/a; q)_n}{(aq; q)_n} \begin{cases} -a \frac{(aq; q)_n}{(q/a; q)_n} \frac{(q/ad, d/a; q^2)_{(n+1)/2}}{(ad, qa/d; q^2)_{(n+1)/2}}, & n \text{ odd} \\ \frac{(aq; q)_n}{(q/a; q)_n} \frac{(q^2/ad, dq/a; q^2)_{n/2}}{(adq, q^2a/d; q^2)_{n/2}}, & n \text{ even,} \end{cases} \\
&= \beta_n^{(2)}(a, q).
\end{aligned}$$

The third equality follows from a  $q$ -analogue of Whipple's  ${}_3F_2$  sum,

$$\begin{aligned}
(3.4) \quad {}_8\phi_7 \left[ \begin{matrix} C, q\sqrt{C}, -q\sqrt{C}, a, q/a, -C, d, q/d \\ \sqrt{C}, -\sqrt{C}, Cq/a, aC, -q, Cq/d, Cd \end{matrix}; q, -C \right] \\
= \frac{(C, Cq; q)_\infty (aCd, aCq/d, Cdq/a, Cq^2/ad; q^2)_\infty}{(Cd, Cq/d, aC, Cq/a; q)_\infty},
\end{aligned}$$

and upon setting  $a = q^{-n}$ ,

$$\begin{aligned}
& {}_8\phi_7 \left[ \begin{matrix} C, q\sqrt{C}, -q\sqrt{C}, d, q/d, -C, q^{-n}, q^{1+n} \\ \sqrt{C}, -\sqrt{C}, Cq/d, dC, -q, Cq^{1+n}, Cq^{-n} \end{matrix}; q, -C \right] \\
&= \begin{cases} \frac{(Cq; q)_n}{(q/C; q)_n} \frac{(q^2/dC, dq/C; q^2)_{n/2}}{(dCq, Cq^2/d; q^2)_{n/2}}, & n \text{ even} \\ \frac{(Cq; q)_n}{(q/C; q)_n} \frac{(q/dC, d/C; q^2)_{(n+1)/2}}{(dC, Cq/d; q^2)_{(n+1)/2}} (-C), & n \text{ odd.} \end{cases}
\end{aligned}$$

□

We had initially thought that the following WP-Bailey pair was totally new also.

$$\begin{aligned}
(3.5) \quad \alpha_n^{(3)}(a, k) &= \frac{(a, q\sqrt{a}, -q\sqrt{a}, k/a, a\sqrt{q/k}, -a\sqrt{q/k}; q)_n}{(\sqrt{a}, -\sqrt{a}, qa^2/k, \sqrt{qk}, -\sqrt{qk}, q; q)_n} (-1)^n, \\
\beta_n^{(3)}(a, k) &= \begin{cases} \frac{(k, k^2/a^2; q^2)_{n/2}}{(q^2, q^2a^2/k; q^2)_{n/2}}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}
\end{aligned}$$

However, this pair can be derived from an existing WP-Bailey pair using a result of Warnaar [14].

**Lemma 3** (Warnaar, [14]). *For  $a$  and  $k$  indeterminates the following equations are equivalent:*

(3.6)

$$\begin{aligned}\beta_n(a, k; q) &= \sum_{j=0}^n \frac{(k/a)_{n-j} (k)_{n+j}}{(q)_{n-j} (aq)_{n+j}} \alpha_j(a, k; q), \\ \alpha_n(a, k; q) &= \frac{1 - aq^{2n}}{1 - a} \sum_{j=0}^n \frac{1 - kq^{2n}}{1 - k} \frac{(a/k)_{n-j} (a)_{n+j}}{(q)_{n-j} (kq)_{n+j}} \left(\frac{k}{a}\right)^{n-j} \beta_j(a, k; q).\end{aligned}$$

If  $a$  and  $k$  are interchanged in the second equation, we see that Lemma 3 implies the following.

**Corollary 9.** *If  $(\alpha_n(a, k), \beta_n(a, k))$  are a WP-Bailey pair, then so are  $(\alpha'_n(a, k), \beta'_n(a, k))$ , where*

$$\begin{aligned}\alpha'_n(a, k) &= \frac{1 - aq^{2n}}{1 - a} \left(\frac{k}{a}\right)^n \beta_n(k, a), \\ \beta'_n(a, k) &= \frac{1 - k}{1 - kq^{2n}} \left(\frac{k}{a}\right)^n \alpha_n(k, a).\end{aligned}$$

For the present purposes, we may call the pair  $(\alpha'_n(a, k), \beta'_n(a, k))$  the *dual* of the pair  $(\alpha_n(a, k), \beta_n(a, k))$ . Thus, for example, our pair at (3.5) is the dual of the WP-Bailey pair of Andrews and Berkovich in [2]:

$$\begin{aligned}\alpha_n(a, k) &= \begin{cases} \frac{(a, q^2\sqrt{a}, -q^2\sqrt{a}, a^2/k^2; q^2)_{n/2}}{(q^2, \sqrt{a}, -\sqrt{a}, q^2k^2/a; q^2)_{n/2}} \left(\frac{k}{a}\right)^n, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \\ \beta_n(a, k) &= \frac{(k, k\sqrt{q/a}, -k\sqrt{q/a}, a/k; q)_n}{(\sqrt{aq}, -\sqrt{aq}, qk^2/a, q; q)_n} \left(\frac{-k}{a}\right)^n.\end{aligned}$$

We had initially thought that the WP-Bailey pair in Lemma 1 was the dual of the following pair of Andrews and Berkovich:

$$\begin{aligned}(3.7) \quad \alpha_n^{(4)}(a, k) &= \frac{1 - aq^{2n}}{1 - a} \frac{(a, k/aq; q)_n}{(q, q^2a^2/k; q)_n} \frac{(qa^2/k; q)_{2n}}{(k; q)_{2n}} \left(\frac{k}{a}\right)^n, \\ \beta_n^{(4)}(a, q) &= \frac{(k^2/qa^2; q)_n}{(q; q)_n}.\end{aligned}$$

However, the dual of the pair in Lemma 1 is the pair

$$\begin{aligned}(3.8) \quad \alpha_n^{(5)}(a, k) &= \frac{1 - aq^{2n}}{1 - a} \frac{(a, kq/a; q)_n}{(q, a^2/k; q)_n} \frac{(a^2/k; q)_{2n}}{(kq; q)_{2n}} \left(\frac{k}{a}\right)^n, \\ \beta_n^{(5)}(a, q) &= \frac{1 - k}{1 - kq^{2n}} \frac{(qk^2/a^2; q)_n}{(q; q)_n},\end{aligned}$$

while the dual of the pair at (3.7) is the pair

$$(3.9) \quad \begin{aligned} \alpha_n^{(6)}(a, k) &= \frac{1 - aq^{2n}}{1 - a} \frac{(a^2/qk^2; q)_n}{(q, q)_n} \left(\frac{k}{a}\right)^n, \\ \beta_n^{(6)}(a, k) &= \frac{(a/kq, k; q)_n}{(k^2q^2/a, q; q)_n} \frac{(qk^2/a; q)_{2n}}{(a; q)_{2n}}. \end{aligned}$$

Inserting the new WP-Bailey pairs in any of the existing WP-Bailey chains will lead to transformations relating basic hypergeometric series. We believe the following transformations of basic hypergeometric series to be new.

**Corollary 10.**

$$(3.10) \quad \begin{aligned} & {}_{12}\phi_{11} \left[ \begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, y, z, \frac{qa}{k}, \frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}, -k\sqrt{\frac{q}{a}}, \frac{kaq^{N+1}}{yz}, q^{-N} \\ \sqrt{k}, -\sqrt{k}, \frac{qk}{y}, \frac{qk}{z}, \frac{k^2}{a}, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}, kq^{N+1}, \frac{yzq^{-N}}{a} \end{matrix} ; q, q \right] \\ &= \frac{(kq, kq/yz, aq/y, aq/z; q)_N}{(kq/y, kq/z, aq/yz, aq; q)_N} {}_5\phi_4 \left[ \begin{matrix} y, z, \frac{qa^2}{k^2}, \frac{kaq^{N+1}}{yz}, q^{-N} \\ \frac{qa}{y}, \frac{qa}{z}, aq^{N+1}, \frac{yzq^{-N}}{k} \end{matrix} ; q, q \right]. \end{aligned}$$

*Proof.* Insert the WP-Bailey pair at (3.1) into (2.3) and set  $\rho_1 = y$  and  $\rho_2 = z$ .  $\square$

Substituting the pair at (3.1) into (2.4) leads to the following result.

**Corollary 11.**

$$(3.11) \quad {}_7\phi_6 \left[ \begin{matrix} k, \frac{qa}{k}, \frac{k}{\sqrt{a}}, -\frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}, -k\sqrt{\frac{q}{a}}, q^{-N} \\ \frac{k^2}{a}, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}, \frac{k^2q^{-N}}{a^2} \end{matrix} ; q, q \right] =$$

$$\frac{(qa/k, qa^2/k; q)_N}{(qa, qa^2/k^2; q)_N} {}_7\phi_6 \left[ \begin{matrix} \frac{qa^2}{k^2}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a^2q^{N+1}}{k}, q^{-N} \\ a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, \frac{kq^{-N}}{a}, aq^{N+1} \end{matrix} ; q, q \right].$$

**Corollary 12.**

$$(3.12) \quad \begin{aligned} & {}_{14}W_{13} \left( q; y, yq, z, zq, \frac{q^2}{ad}, \frac{dq}{a}, q^2, \frac{aq^{N+2}}{yz}, \frac{aq^{N+3}}{yz}, q^{-N}, q^{1-N}; q^2, q^2 \right) - \\ & \frac{aq \left(1 - \frac{d}{a}\right) \left(1 - \frac{q}{ad}\right) (1 - q^3) (1 - q^{-N}) (1 - y) \left(1 - \frac{aq^{2+N}}{yz}\right) (1 - z)}{(1 - ad) (1 - q) \left(1 - \frac{aq}{d}\right) (1 - q^{2+N}) \left(1 - \frac{q^2}{y}\right) \left(1 - \frac{q^2}{z}\right) \left(1 - \frac{yzq^{-N}}{a}\right)} \end{aligned}$$

$$\begin{aligned}
& \times {}_{14}W_{13} \left( q^3; yq, yq^2, zq, zq^3, \frac{q^3}{ad}, \frac{dq^2}{a}, q^2, \frac{aq^{N+3}}{yz}, \frac{aq^{N+4}}{yz}, q^{1-N}, q^{2-N}; q^2, q^2 \right) \\
& = \frac{\left( q^2, \frac{q^2}{yz}, \frac{aq}{y}, \frac{aq}{z}; q \right)_N}{\left( \frac{q^2}{y}, \frac{q^2}{z}, \frac{aq}{yz}, aq; q \right)_N} {}_{10}W_9 \left( a; y, z, d, \frac{q}{d}, -a, \frac{aq^{N+2}}{yz}, q^{-N}; q, -a \right). \\
(3.13) \quad & {}_7\phi_6 \left[ \begin{matrix} q^{5/2}, -q^{5/2}, y, \frac{q}{y}, \frac{q^2}{ad}, \frac{dq}{a}, q^2 \\ q^{1/2}, -q^{1/2}, q^2y, \frac{q^3}{y}, qad, \frac{q^2a}{d} \end{matrix}; q^2, a^2 \right] \\
& - \frac{a^2 \left(1 - \frac{d}{a}\right) \left(1 - \frac{q}{ad}\right) (1 - q^3) \left(1 - \frac{q}{y}\right) (1 - y)}{(1 - ad) (1 - q) \left(1 - \frac{aq}{d}\right) \left(1 - \frac{q^2}{y}\right) (1 - qy)} \\
& \times {}_7\phi_6 \left[ \begin{matrix} q^{7/2}, -q^{7/2}, yq, \frac{q^2}{y}, \frac{q^3}{ad}, \frac{dq^2}{a}, q^2 \\ q^{3/2}, -q^{3/2}, q^3y, \frac{q^4}{y}, q^2ad, \frac{q^3a}{d} \end{matrix}; q^2, a^2 \right] \\
& = \frac{(q, q^2; q)_\infty (ady, adq/y, ayq/d, q^2a/dy; q^2)_\infty}{(ab, aq/d, yq, q^2/y; q)_\infty}.
\end{aligned}$$

*Proof.* Insert the WP-Bailey pair at (3.3) into (2.3), set  $\rho_1 = y$  and  $\rho_2 = z$  and replace  $k$  with  $q$ . For (3.13), set  $z = q/y$ , let  $N \rightarrow \infty$  and use (3.4) to sum the resulting right side.  $\square$

**Corollary 13.**

$$\begin{aligned}
(3.14) \quad & {}_5\phi_4 \left[ \begin{matrix} \frac{q^2}{ad}, \frac{dq}{a}, q^{-N}, q^{1-N}, q^2 \\ \frac{aq^2}{d}, adq, \frac{q^{2-N}}{a^2}, \frac{q^{3-N}}{a^2} \end{matrix}; q^2, q^2 \right] \\
& - aq \frac{(1 - q^{-N})(1 - q/ad)(1 - d/a)}{(1 - q^{2-N}/a^2)(1 - ad)(1 - aq/d)} {}_5\phi_4 \left[ \begin{matrix} \frac{q^3}{ad}, \frac{dq^2}{a}, q^{1-N}, q^{2-N}, q^2 \\ \frac{aq^3}{d}, adq^2, \frac{q^{3-N}}{a^2}, \frac{q^{4-N}}{a^2} \end{matrix}; q^2, q^2 \right] \\
& = \frac{(a, a^2; q)_N}{(qa, a^2/q; q)_N} {}_{10}W_9 \left( a; \sqrt{q}, -\sqrt{q}, d, \frac{q}{d}, q, a^2q^N, q^{-N}; q, -a \right).
\end{aligned}$$

*Proof.* Insert the WP-Bailey pair at (3.3) into (2.4), and replace  $k$  with  $q$ .  $\square$

Remark: The extra  $q$ -products inserted in each of the series on the right side of (3.12) and in (3.14) are there to allow these series to be represented as  ${}_{r+1}\phi_r$  or  ${}_{r+1}W_r$  series.

**Corollary 14.**

$$\begin{aligned}
 (3.15) \quad {}_{12}W_{11} & \left( k; \frac{qa}{k}, \frac{k}{\sqrt{a}}, \frac{-k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}, -k\sqrt{\frac{q}{a}}, \sqrt{ak}q^N, -\sqrt{ak}q^N, -q^{-N}, q^{-N}; q, q^2 \right) \\
 &= \frac{\left(\frac{k}{a}, k^2q^2, -a, -aq; q^2\right)_N}{\left(\frac{a}{k}, a^2q^2, -kq, -kq^2; q^2\right)_N} \left(\frac{aq}{k}\right)^N \\
 & \times {}_7\phi_6 \left( \begin{matrix} iq\sqrt{a}, -iq\sqrt{a}, \frac{a^2q}{k}, \sqrt{ak}q^N, -\sqrt{ak}q^N, -q^{-N}, q^{-N} \\ i\sqrt{a}, -i\sqrt{a}, aq^{1+N}, -aq^{1+N}, \sqrt{\frac{a}{k}}q^{1-N}, -\sqrt{\frac{a}{k}}q^{1-N} \end{matrix}; q, q \right).
 \end{aligned}$$

*Proof.* After replacing  $m$  with  $k$  and employing some simple transformations for  $q$ -products, (2.15) can be rewritten as

$$\begin{aligned}
 (3.16) \quad \sum_{j=0}^N \frac{1 - kq^{2j}}{1 - k} \frac{(akq^{2N}, q^{-2N}; q^2)_j}{(k^2q^{2+2N}, \frac{k}{a}q^{2-2N}; q^2)_j} q^{2j} \beta_j(a, k) \\
 = \frac{\left(\frac{k}{a}, k^2q^2; q^2\right)_N}{\left(\frac{a}{k}, a^2q^2; q^2\right)_N} \frac{(-a; q)_{2N}}{(-kq; q)_{2N}} \left(\frac{aq}{k}\right)^N \\
 \times \sum_{j=0}^N \frac{1 + aq^{2j}}{1 + a} \frac{(akq^{2N}, q^{-2N}; q^2)_j}{(a^2q^{2+2N}, \frac{a}{k}q^{2-2N}; q^2)_j} \left(\frac{aq}{k}\right)^j \alpha_j(a, k).
 \end{aligned}$$

The result follows, upon inserting the pair from Lemma 1, and rearranging.  $\square$

Inserting the new pairs in other WP-Bailey chains will lead to other, possibly new, transformations of basic hypergeometric series, but we refrain from further examples here.

#### 4. WP-BURGE PAIRS

In [2], the authors termed a WP-Bailey pair  $(\alpha_n(1, k), \beta_n(1, k))$  in which  $\alpha_n(1, k)$  does not depend on  $k$  a *WP-Burge pair*.

One such pair that they derive in [2, (7.16)] is the pair in the following theorem. For completeness, we give an alternative proof of this result.

**Theorem 2.** *Define*

$$\begin{aligned}
 (4.1) \quad \alpha_n(1, k) &= \begin{cases} 1, & n = 0, \\ q^{-n/2} + q^{n/2}, & n \geq 1, \end{cases} \\
 \beta_n(1, k) &= \frac{(k\sqrt{q}, k; q)_n}{(\sqrt{q}, q; q)_n} q^{-n/2}.
 \end{aligned}$$

*Then  $(\alpha_n(1, k), \beta_n(1, k))$  satisfy (1.1) (with  $a = 1$ ).*

*Proof.* We begin by recalling Bailey's  ${}_6\psi_6$  summation formula [13].

$$\begin{aligned}
& \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty} \\
&= \sum_{n=-\infty}^{\infty} \frac{(1 - aq^{2n})(b, c, d, e; q)_n}{(1 - a)(aq/b, aq/c, aq/d, aq/e; q)_n} \left( \frac{qa^2}{bcde} \right)^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(1 - aq^{2n})(b, c, d, e; q)_n}{(1 - a)(aq/b, aq/c, aq/d, aq/e; q)_n} \left( \frac{qa^2}{bcde} \right)^n \\
&\quad + \sum_{n=1}^{\infty} \frac{(1 - 1/aq^{2n})(b/a, c/a, d/a, e/a; q)_n}{(1 - 1/a)(q/b, q/c, q/d, q/e; q)_n} \left( \frac{qa^2}{bcde} \right)^n,
\end{aligned}$$

where the second equality follows from the definition

$$(z; q)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{z^n (q/z; q)_n}.$$

Next, set  $a = -1$ ,  $b = -c$  and  $d = -e$ , so that both sums in the final expression above become equal, to get

$$\begin{aligned}
& \frac{(-q, q/c^2, q/e^2; q)_\infty (q^2/c^2e^2, q^2/c^2e^2, q^2; q^2)_\infty}{(q/c^2e^2; q)_\infty (q^2/c^2, q^2/c^2, q^2/e^2, q^2/e^2; q^2)_\infty} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(1 + q^{2n})(c^2, e^2; q^2)_n}{(q^2/c^2, q^2/e^2; q^2)_n} \left( \frac{q}{c^2e^2} \right)^n,
\end{aligned}$$

or, upon replacing  $c^2$  with  $c$  and  $e^2$  with  $e$ ,

$$\begin{aligned}
(4.2) \quad & \frac{(-q, q/c, q/e; q)_\infty (q^2/ce, q^2/ce, q^2; q^2)_\infty}{(q/ce; q)_\infty (q^2/c, q^2/c, q^2/e, q^2/e; q^2)_\infty} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(1 + q^{2n})(c, e; q^2)_n}{(q^2/c, q^2/e; q^2)_n} \left( \frac{q}{ce} \right)^n.
\end{aligned}$$

Now replace  $q$  with  $\sqrt{q}$ , set  $e = q^{-N}$ ,  $c = kq^N$  to get, after some minor rearrangements, that

$$1 + \sum_{n=1}^N \frac{(kq^N, q^{-N}; q)_n}{(q^{1-N}/k, q^{1+N}; q)_n} \left( \frac{q}{k} \right)^n (q^{-n/2} + q^{n/2}) = \frac{(k\sqrt{q}, q; q)_N}{(k, \sqrt{q}; q)_N} q^{-N/2}$$

The result now follows from (1.1), after some simple manipulations.  $\square$

One reason this WP-Bailey pair is somewhat interesting is that the special case  $k = 0$  of (4.1) gives Slater's standard Bailey pair **F3** from [10]. Recall that a pair of sequences  $(\alpha_n, \beta_n)$  is termed a *Bailey pair relative to  $x/q$* , if they satisfy

$$(4.3) \quad \beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (x; q)_{n+r}},$$

We also recall the following result of Bailey, a particular case of the ‘‘Bailey Transform’’, from his 1949 paper [4].

**Theorem 3.** *Subject to suitable convergence conditions, if the sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  satisfy (4.3), then*

$$(4.4) \quad \sum_{n=0}^{\infty} (y, z; q)_n \left( \frac{x}{yz} \right)^n \beta_n = \frac{(x/y, x/z; q)_{\infty}}{(x, x/yz; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(x/y, x/z; q)_n} \left( \frac{x}{yz} \right)^n \alpha_n.$$

Thus the pair at (4.1) “lifts” this standard Bailey pair to a WP-Bailey pair and lifts all the series–product identities following from **F3** to more general series–product identities containing the free parameter  $k$ .

More generally, let  $(\alpha_n(a, k), \beta_n(a, k))$  be a WP-Bailey pair in which  $\alpha_n(a, k)$  is independent of  $k$ . Suppose further that this pair is a “lift” of a standard Bailey pair, in the sense that setting  $k = 0$  in the WP-Bailey pair recovers the Bailey pair. By comparing (4.4) and (2.5) (first setting  $y = \rho_1$ ,  $z = \rho_2$  and  $x = aq$  in (4.4)) we see that the two infinite series containing  $\alpha_n = \alpha_n(a, k)$  are identical. This means that if particular choices of  $y$  and  $z$  in (4.4) lead to an identity of the Rogers-Ramanujan type, then the same choices for  $\rho_1$  and  $\rho_2$  in (2.5) will lead to a more general series–product identity containing an extra free parameter, namely  $k$ , and this more general identity will revert back to the original identity of Rogers-Ramanujan type, upon setting  $k = 0$ .

We will investigate this phenomenon further in a subsequent paper.

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